Math 249 Lecture 5 Notes

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1 G-Modules

1.1 G-Modules

Recall from last time that we defined matrix representations $G \to \mathrm{GL}_n(V)$ and linear representations $G \to \mathrm{GL}(V)$. We sometimes also call the latter a G-module V, and say that $G \circlearrowright V$.

Definition 1.1. A *homomorphism* of G modules $V \to W$ is a linear map φ such that $g \cdot \varphi(v) = \varphi(g \cdot v)$ for every $g \in G$ and $v \in V$.

Definition 1.2. An *isomorphism* of G modules is a bijective homomorphism of G-modules.

If V is finite dimensional (dimension n), pick basis $\{v_1, \dots, v_n\}$. Then there is an isomorphism $\varphi : \mathbb{C}^n \to V$ sending $e_i \mapsto v_i$. Define $\rho : G \circlearrowright \mathbb{C}^n$ as $\rho(g) \cdot x = \varphi^{-1}(g \cdot \varphi(x))$; this makes φ an isomorphism of G modules. In this way, we can make a linear representation with a basis into a matrix representation.

1.2 Submodules

Definition 1.3. Let V be a G-module. A *submodule* is a G-invariant subspace $W \subseteq V$.

If W is a submodule of V, V/W is a G-module, where $g \cdot (v + W) = g \cdot v + W$.

Look at this from the perspective of matrix representations. Pick a basis $\{v_1, \dots, v_n\}$ of V such that $\{v_1, \dots, v_k\}$ is a basis of W. Then our representation ρ sends g to the block matrix

$$g \mapsto \begin{pmatrix} \rho_W(g) & * \\ 0 & \rho_{V/W}(g), \end{pmatrix}.$$

where the sizes of the blocks are $k \times k$, $k \times (n-k)$, $(n-k) \times k$, and $(n-k) \times (n-k)$. The unknown part can be 0. In fact, if $W_1, W_2 \subseteq V$ are submodules such that $W_1 \oplus W_2 = V$ (V is a direct sum of G-modules), then we have

$$g \mapsto \begin{pmatrix} \rho_{W_1}(g) & 0 \\ 0 & \rho_{W_2}(g) \end{pmatrix}.$$

V has a character $\chi_V(g) = \mathrm{tr}(g, V)$, where we can determine the trace by picking any basis. Then $\chi_{W_1 \oplus W_2} = \chi_{W_1} + \chi_{W_2}$.

Given $G \circlearrowleft X$, we can define an action $G \circlearrowleft \mathbb{C}X$, where $\mathbb{C}X$ is the vector space over $\mathbb{C}$ with basis X; extend the action on the basis elements linearly. This can be viewed as sending $G \to S(X)$, the symmetric group on X, and acting each P_σ on $\mathbb{C}X$.

If $W_1 = \mathbb{C}\cdot(1, \ldots, 1)$, it is a trivial submodule. $V/W_1 = \mathbb{C}^n/(\mathbb{C}\cdot(1, \ldots, 1))$ has dimension $n-1$. Let $W_2 = \{(x_1, \ldots, x_n) : \sum x_i = 0\}$. Then $V = W_1 \oplus W_2$. In fact, $W_2 \cong V/W_1$.

Remark 1.1. This does not hold in other fields in general. Let K be a field with characteristic p dividing n, and let $x = c \cdot (1, \ldots, 1)$. Then $\sum x_i = nc = 0 \pmod p$. So if p divides n, then $0 \subseteq W_1 \subseteq W_2 \subseteq V$, and the direct sum picture does not work out the same.

1.3 The group algebra

Definition 1.4. Let G be a group. The *group algebra* $\mathbb{C}G$ is the vector space over $\mathbb{C}$ with basis G. This is formal linear combinations of elements of G with coefficients in $\mathbb{C}$. The multiplication is determined by

$$\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g b_h gh,$$

where gh is defined by the group multiplication of G.

A G-module is a left $\mathbb{C}G$-module. If V is a left $\mathbb{C}G$-module, then $\mathbb{C} = \mathbb{C} \cdot 1 \subseteq \mathbb{C}G$, where 1 is the group identity. This makes V a $\mathbb{C}$ vector space. Additionally, $G \subseteq \mathbb{C}G$, which makes $\mathbb{C}G \circlearrowleft V$ (by linear maps); i.e. $(\sum_g a_g g) \cdot v = \sum_g a_g (g \cdot v)$.